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A QUESTION OF PASSMAN ON THE SYMMETRIC RING OF QUOTIENTS†

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ABSTRACT

We give an example of a domain such that the sequence of iterated symmetric rings of quotients does not stabilize. This answers a question of Passman.

The symmetric ring of quotients of a prime ring R was introduced and studied by Passman [4]. It can be described as the set of elements q of the right Martindale ring of quotients of R such that $Iq \subset R$ for some nonzero (two-sided) ideal I of R. This ring was used by Kharchenko in his investigations on the Galois theory of semiprime rings, see [2].

Denote the symmetric ring of quotients of a prime ring R by $Q_s(R)$. Set $Q_0(R) = R$, $Q_i(R) = Q_s(Q_{i-1}(R))$ for $i \geqq 1$. In [3] Lewin shows that if R is a 2-fir then $Q_1(R)$ is symmetrically closed, so $Q_n(R) = Q_1(R)$ for all $n \geqq 1$. On the other hand, Passman shows in [4] that the Bergman's ring $R = K[t][x, y \mid xy = tyx]$, first introduced in [1], satisfies $Q_1(R) \neq Q_2(R)$ and $Q_n(R) = Q_2(R)$ for all $n \geqq 2$.

In this paper we shall construct an example of a domain R such that $Q_i(R)$ is strictly contained in $Q_{i+1}(R)$ for all i. This answers a question of Passman [4, Question 2]. Our example is an adaptation of Bergman's one.

Let K be a field with a nonzero element $\omega \in K$ such that $\omega^n \neq 1$ for all $n \neq 0$. We will denote by $A = K\langle y_1, y_2, \dots \rangle$ the free algebra in the noncommuting indeterminates $y_1, y_2, \dots$.

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It is a well known result that if R is a domain and $\sigma: R \to R$ is an injective ring endomorphism then $S = R[x; \sigma]$ (i.e. S is the ring of polynomials in one variable with coefficients in R with the classical additive structure and the multiplicative structure given by $rx = xr^{\sigma}$) is also a domain.

We will construct a chain of rings $A = A_0 \subset A_1 \subset A_2 \subset \cdots$ in the following way: $A_1 = A[x_1; \sigma_1]$ where $\sigma_1 : A \to A$ is the algebra endomorphism such that it is the identity on K, $y_n^{\sigma_1} = y_n$ if $n \neq 1$ and $y_1^{\sigma_1} = \omega y_1$. Assume that we have constructed $A_n = A[x_1, x_2, \ldots, x_n; \sigma_1, \sigma_2, \ldots, \sigma_n]$. Then we define $A_{n+1} = A_n[x_{n+1}; \sigma_{n+1}]$ where $\sigma_{n+1} : A_n \to A_n$ is the identity on $K[x_1, x_2, \ldots, x_n]$, $y_m^{\sigma_{n+1}} = y_m$ if $m \neq n+1$ and $y_{n+1}^{\sigma_{n+1}} = x_n y_{n+1}$. It is an easy exercise to check that σ_{n+1} defines an injective ring endomorphism on A_n and hence A_{n+1} is a domain.

Let $R = \bigcup_{n \geqq 0} A_n$. Since any A_n is a domain, also R is a domain. Moreover, R can be described as the algebra $K[x_1, y_1, x_2, y_2, \ldots]$ with relations:

(a) x_i commutes with x_j, y_j if $i \neq j$,

(b) $y_1 x_1 = \omega x_1 y_1$,

(c) $y_n x_n = x_{n-1} x_n y_n$ for $n > 1$.

Set $B = K[x_1, x_2, \ldots]$ the polynomial ring in the commuting indeterminates $x_1, x_2, \ldots$ and S the free semigroup with 1 generated by $y_1, y_2, \ldots$. Then R is a free left B-module with free basis S.

If $s \in S$ and $\alpha \in B$, then there is a unique $H_s(\alpha) \in B$ such that $s\alpha = H_s(\alpha)s$. Moreover, for every $s \in S$, H_s is an algebra endomorphism of B. On the other hand $s \mapsto H_s$ is a multiplicative map between S and $\mathrm{End}_K(B)$. If α is a monomial in B and $s \in S$, then $H_s(\alpha) = \alpha \varphi_s(\alpha)$ for some monomial $\varphi_s(\alpha) \in B$. Also it is trivial that $\varphi_s(x_1) \in K$ and for every $n > 1$, $\varphi_s(x_n) \in K[x_1, x_2, \ldots, x_{n-1}]$. Moreover, the degree in x_{n-1} of $\varphi_s(x_n)$ is the number of appearances of y_n in s.

Lemma 1. *Let I be a nonzero ideal in R. Then there exists $0 \neq \alpha \geqq \sum_{s \in S} \alpha_s s \in I$ with $\alpha_s \in B$ for every $s \in S$, such that $\alpha_t \neq 0$ is a monomial for some $t \in S$.*

Proof. Assume that for every $\alpha = \sum_{s \in S} \alpha_s s \in I$ any nonzero α_s is not a monomial. Then for every such α_s there exist unique $p, k \geqq 1$ so that $\alpha_s = m \sum_{i=0}^{p} x_k^i \beta_i$ with $\beta_i \in K[x_1, x_2, \ldots, x_{k-1}]$, $\beta_0, \beta_p \neq 0$ and $m \in B$ is a monomial. Choose $\alpha \in I$ with a coefficient α_t with k minimal and p minimal among all the elements with k minimal. Consider the element

$$\alpha' = \beta_0 t y_k \alpha - \varphi_{ty_k}(m) H_t(\beta_0) \alpha y_k t \in I.$$

Then $\alpha' = \alpha'_t ty_k t + \Sigma_{s \neq t} \delta_s sy_k t + \Sigma_{s \neq t} \gamma_s ty_k s$ with α'_t, δ_s, $\gamma_s \in B$ for every $s \in S$. If either $ty_k t = ty_k s$ or $ty_k t = sy_k t$ then $t = s$. So the coefficient of $ty_k t$ in α' is exactly α'_t. Now we have

$$\alpha'_t = m\varphi_{ty_k}(m) \sum_{i=0}^{p} x_k^i \gamma_i$$

where

$$\gamma_i = \varphi_{ty_k}(x_k^i) H_t(\beta_i)\beta_0 - H_t(\beta_0)\beta_i$$

for $0 \leqq i \leqq p$. Observe that $\gamma_0 = 0$ and $\gamma_i \in K[x_1, \ldots, x_{n-1}]$. So we have

$$\alpha'_t = m' \sum_{i=0}^{p-1} x_k^i \gamma_{i+1}$$

where $m' = m\varphi_{ty_k}(m)x_k$ is a monomial. If we prove that $\gamma_p \neq 0$ then we will arrive at a contradiction with the choice of α_t.

First assume that $k > 1$. In this case

$$\gamma_p = x_{k-1}^p \varphi_t(x_k^p x_{k-1}^p) H_t(\beta_p)\beta_0 - H_t(\beta_0)\beta_p.$$

Observe that the degrees in x_{k-1} of $H_t(\beta_p)\beta_0$ and $H_t(\beta_0)\beta_p$ are equal to the degree in x_{k-1} of $\beta_0\beta_p$, since $\beta_0, \beta_p \in K[x_1, x_2, \ldots, x_{p-1}]$. It follows that the degree in x_{k-1} of $x_{k-1}^p \varphi_t(x_k^p x_{k-1}^p) H_t(\beta_p)\beta_0$ is strictly larger than that of $H_t(\beta_0)\beta_p$ and consequently $\gamma_p \neq 0$.

Now we assume that $k = 1$. In this case we have $\gamma_p = (\omega^{p(n+1)} - 1)\beta_0\beta_p$ where n is the number of appearances of y_1 in s. Since ω is not a root of the unity we see that $\gamma_p \neq 0$. ■

Let M be the free commutative semigroup with 1 generated by $x_1, x_2, \ldots$. Clearly M can be viewed as a multiplicative subset of R and it is a two-sided Ore set. So we can define $Q = RM^{-1}$, the corresponding Ore localization. Write $D = K[x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots]$. Then Q is a free left and right D-module with basis S. If $x = \Sigma_{s \in S} \lambda_s s \in Q$ ($\lambda_s \in D$ for every $s \in S$), we define the *support* of x as $\mathrm{Supp}(x) = \{s \in S \mid \lambda_s \neq 0\}$. Obviously the functions H_s (resp. φ_s) can be extended to D (resp. monomials in D).

As in [4] we say that a subset $V \subset S - \{1\}$ is *separated* if for all $a, b \in V$, if $w \neq 1$ is an initial segment of a and a final segment of b, then we must have $a = w = b$. We will need the following easy lemma.

LEMMA 2. *Let T be a ring such that $R \subset T \subset Q$ and let V be a finite separated subset of S. If $\Sigma_{a \in V} \beta_a a = 0$ where $\beta_a \in Q_s(T)$ then $\beta_a = 0$ for all $a \in V$.*

PROOF. There exists a nonzero ideal I of T such that $I\beta_a \subset T$ for all $a \in V$. Choose a nonzero element x in I. Then $\Sigma_{a\in V}(x\beta_a)a = 0$. Since V is separated, V is a left Q-independent family by [4, Lemma 2.2(ii)]. It follows that $x\beta_a = 0$ for all $a \in V$ and, since $Q_s(T)$ is a domain [4, Lemma 1.7], $\beta_a = 0$ for all $a \in V$. ∎

The proof of the following Proposition follows the lines of that of [4, Theorem 2.5].

PROPOSITION 3. *Let T be a ring such that $R \subset T \subset Q$. Then $Q_s(T) \subset Q$. In particular, Q is symmetrically closed.*

PROOF. Let q be an element in $Q_s(T)$. There exists a nonzero ideal I_0 of T such that $qI_0 \subset T$ and $I_0 q \subset T$. Set $J = \{x \in T \mid 1 \notin \mathrm{Supp}(x)\}$. Then J is a nonzero ideal of T, and putting $I = JI_0J$, we have $qI \subset J$ and $Iq \subset J$. By Lemma 1 there exists $\alpha = \Sigma_{s\in V}\alpha_s s \in I \cap R$ such that $\alpha_t \in M$ for some $t \in V = \mathrm{Supp}(\alpha)$. By using [4, Lemma 2.3] we see that we can assume that V is a separated subset of S. Set $\beta = q\alpha \in T$ and $\gamma = \alpha q \in T$. Then we have $\alpha\beta = \alpha q \alpha = \gamma\alpha$. Since $\mathrm{Supp}(\alpha)$ is separated we obtain, as in the proof of [4, Lemma 2.4], that $\beta = \Sigma_{s\in V}\tau_s s$ with $\tau_s \in Q$. There exists $m \in M$ such that $m\tau_s \in R$ for all $s \in V$. Hence

$$m\beta = \sum_{s\in V} m\tau_s s = mq\alpha = \sum_{s\in V}(mq\alpha_s)s.$$

It follows that $\Sigma_{s\in V}(m\tau_s - mq\alpha_s)s = 0$ and so by Lemma 2 we have $m\tau_s = mq\alpha_s$ for $s \in V$. Now we view $Q_s(T)$ and Q as subrings of the maximal quotient ring of T and in this ring we have $\tau_s = q\alpha_s$ for all $s \in V$. In particular $\tau_t = q\alpha_t$ and so $q = \tau_t\alpha_t^{-1} \in Q$. ∎

Define $R_0 = R$, $R_n = RM_n^{-1}$ where M_n is the subsemigroup of M generated by $x_1, x_2, \dots, x_n$. Then $R_0 \subset R_1 \subset \cdots$ and $Q = \bigcup_{i=0}^{\infty} R_i$. For every $i = 1, 2, \dots, R_i$ is a free left B_i-module where $B_i = K[x_1, x_1^{-1}, \dots, x_i, x_i^{-1}, x_{i+1}, x_{i+2}, \dots]$. On the other hand set $Q_0 = R$, $Q_i = Q_s(Q_{i-1})$ for $i \geqq 1$.

THEOREM 4. *With the above notation, $R_i = Q_i$ for all $i \geqq 0$.*

PROOF. Assume that $R_m = Q_m$ for some $m \geqq 0$. We will show that $R_{m+1} = Q_{m+1}$. Since x_{m+1}^{-1} normalizes R_m we see that $R_{m+1} \subset Q_{m+1}$. Observe also that Proposition 3 implies that $Q_{m+1} \subset Q$.

Let $q = \Sigma_{s\in V}\lambda_s s \in Q_{m+1} \setminus R_{m+1}$ where $V = \mathrm{Supp}(q)$ and $\lambda_s \in D$. For some $t \in V$, $\lambda_t = \Sigma_{j=n}^{l} x_k^j \mu_j$ with $k \geqq m+2$,

$$\mu_j \in K[x_1, x_1^{-1}, \ldots, x_{k-1}, x_{k-1}^{-1}, x_{k+1}, x_{k+1}^{-1}, \ldots], \quad n < 0 \quad \text{and} \quad \mu_n \neq 0.$$

By multiplying q by suitable elements of M and then subtracting some element of R we see that Q_{m+1} contains a nonzero element of the form $p = x_k^{-1}\beta$ with $\beta = \Sigma_{s \in V} \beta_s s$, $\beta_s \in K[x_1, \ldots, x_{k-1}, x_{k+1}, \ldots]$. Let I be a nonzero ideal of R_m such that $Ip \subset R_m$. Let $\alpha = \Sigma_{s \in S} \alpha_s s$ be a nonzero element of I with $\alpha_s \in B_m$. We have $\alpha y_k^r x_k^{-1}\beta \in R_m$ for all $r \geqq 0$. Now

$$\begin{aligned}\alpha y_k^r x_k^{-1}\beta &= \sum_{s \in S} \alpha_s s y_k^r x_k^{-1}\beta \\ &= \sum_{s,t \in S} \alpha_s s x_{k-1}^{-r} x_k^{-1} \beta_t y_k^r t \\ &= x_{k-1}^{-r}\left(\sum_{s,t \in S} \gamma_{sy_k^r t}\, s y_k^r t\right)\end{aligned}$$

where $\gamma_{sy_k^r t} = \alpha_s \varphi_s(x_{k-1}^{-r}) H_s(x_k^{-1}\beta_t)$. Since the degrees in x_{k-1} of $\{\gamma_{sy_k^r t}\}$ are bounded by a constant that does not depend on r and $k-1 \geqq m+1$, we see that, for large r, the element $\alpha y_k^r x_k^{-1}\beta$ does not belong to R_m, which is a contradiction. So $Q_{m+1} = R_{m+1}$. ■

Remarks. (1) We deduce from Theorem 4 that $R_{i+1} = R_i N$, the normal closure of R_i [1], for all $i \geqq 0$. So the iterated normal closure of R does not stabilize.

(2) Let $Q_l^i(R)$ be the i-th iterated left Martindale ring of quotients of R. Then the rings $Q_l^i(R)$ and Q are subrings of the left maximal ring of quotients of R. By using the proof of Theorem 4 we see that $Q_l^i(R) \cap Q = R_i$ for all i. It follows that the sequence $\{Q_l^i(R)\}$ does not stabilize.

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